# On the Local Dirichlet-to-Neumann Map

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**Abstract.** We survey some recent progress on the problem of determining a conductivity or a potential by measuring the elliptic Dirichlet-to-Neumann map for the associated conductivity equation or the Schrödinger equation. We make emphasis on the new results obtained on open problem 2 stated in [21] which concerns with the case that the measurements are made on a strict subset of the boundary.

## 1 Anisotropic Conductivities

My lectures at the EuroSummer School were about the anisotropic inverse conductivity problem. We start this section by describing the problem. We also state and sketch the proof of some recent results of [12] and [13].

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $\gamma = (\gamma^{ij}(x))$ be the electrical conductivity of $\Omega$ which is assumed to be a positive definite, smooth, symmetric matrix on $\overline{\Omega}$. Muscle tissue in the human body is a prime example of an anisotropic conductivity since the conductivity in the transverse direction (for cardiac muscle this is 2.3 mho) is quite different from that of the longitudinal direction (for cardiac muscle this is 6.3 mho).

Under the assumption of no sources or sinks of current in $\Omega$, the equation for the potential, given a voltage potential $f$ on $\partial\Omega$, is given by the solution of the Dirichlet problem

$$\begin{cases} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \text{ on } \Omega \\ u|_{\partial\Omega} = f. \end{cases} \tag{1}$$

The Dirichlet-to-Neumann map (DN) is defined by

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega} \tag{2}$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and $u$ is the solution of (1). $\Lambda_\gamma$ is also called the *voltage* to *current* map since $\Lambda_\gamma(f)$ measures the induced current flux at the boundary.

The inverse problem is whether one can determine $\gamma$ by knowing $\Lambda_\gamma$. Calderón proposed this problem in [5] and obtained the first results in the

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multidimensional case. It arose originally in geophysics [23]. More recently this inverse problem has been proposed as a valuable diagnostic tool in medicine (see for instance [2]) and it has been called *electrical impedance tomography* (EIT). Unfortunately, $\Lambda_\gamma$ doesn't determine $\gamma$ uniquely. This observation is due to L. Tartar (see [11] for an account). To verify this we define first the Dirichlet integral associated to a solution of (1). Let

$$Q_\gamma(f) = \sum_{i,j=1}^{n} \int_\Omega \gamma^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \tag{3}$$

with $u$ a solution of (1).

A standard application of the divergence theorem gives that

$$Q_\gamma(f) = \int_{\partial\Omega} \Lambda_\gamma(f) f dS, \tag{4}$$

where $dS$ denotes surface measure in $\partial\Omega$. In other words, $\Lambda_\gamma$ is the linear operator associated to the quadratic form $Q_\gamma$ so that $\Lambda_\gamma$ and $Q_\gamma$ carry the same information.

Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a $C^\infty$ diffeomorphism with $\psi|_{\partial\Omega} =$ Identity. Let $v = u \circ \psi^{-1}$. Then a straightforward calculation shows that $v$ satisfies

$$\begin{cases} \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \widetilde{\gamma}_{ij} \frac{\partial v}{\partial x_j} \right) = 0 \\ v|_{\partial\Omega} = f \end{cases} \tag{5}$$

where

$$\widetilde{\gamma} = \left( \frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|} \right) \circ \psi^{-1} =: \psi_* \gamma. \tag{6}$$

Here $D\psi$ denotes the (matrix) differential of $\psi$, $(D\psi)^T$ its transpose and the composition in (6) is to be interpreted as composition of matrices.

By making the change of variables $v = u \circ \psi^{-1}$ in the quadratic form (3) we see that

$$Q_{\widetilde{\gamma}}(f) = Q_\gamma(f) \tag{7}$$

and therefore $\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma$.

We have found a large number of conductivities with the same DN map: any change of variables of $\Omega$ that leaves the boundary fixed gives rise to a new conductivity with the same electrical boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity. As we outline below this is a problem of geometrical nature and we proceed to state it in invariant form.

Let $(M, g)$ be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric $g$ is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{\det g} g^{ij} \frac{\partial u}{\partial x_j} \right) \tag{8}$$

where $(g^{ij})$ is the inverse of the metric $g$. Let us consider the Dirichlet problem associated to (1)

$$\Delta_g u = 0 \text{ on } M, \quad u\Big|_{\partial M} = f. \tag{9}$$

We define the DN map in this case by

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^i g^{ij} \frac{\partial u}{\partial x_j} \sqrt{\det g}\Big|_{\partial M} \tag{10}$$

The inverse problem is to recover $g$ from $\Lambda_g$.

By using a similar argument to the one outlined above we have that

$$\Lambda_{\psi^* g} = \Lambda_g \tag{11}$$

where $\psi$ is a $C^\infty$ diffeomorphism of $\overline{M}$ which is the identity on the boundary. As usual $\psi^* g$ denotes the pull back of the metric $g$ by the diffeomorphism $\psi$.

In the case that $M$ is an open, bounded subset of $\mathbb{R}^n$ with smooth boundary, it is easy to see that ([12]) for $n \geq 3$

$$\Lambda_g = \Lambda_\gamma \tag{12}$$

where

$$g_{ij} = (\det \gamma^{kl})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad \gamma^{ij} = (\det g_{kl})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{13}$$

In the two dimensional case (12) is not valid. In fact in $n = 2$ the Laplace-Beltrami operator is conformally invariant. More precisely

$$\Delta_{\alpha g} = \frac{1}{\alpha} \Delta_g$$

for any function $\alpha$, $\alpha \neq 0$. Therefore we have that for $n = 2$

$$\Lambda_{\alpha(\psi^* g)} = \Lambda_g \tag{14}$$

for any smooth function $\alpha \neq 0$ so that $\alpha|_{\partial M} = 1$.

Now we give an invariant formulation of the EIT problem in the two dimensional case. In the Euclidean case a current is a one form given by

$$i(x) = \gamma(x) du(x)$$

where $u$ is the voltage potential. Then, in two dimensions, the conductivity $\gamma$ can be viewed as a linear map from 1-forms to 1-forms. Now let $(M, g)$ be a two dimensional Riemannian manifold. Let $\gamma$ be a positive definite symmetric mapping (with respect to the inner product defined by the metric $g$) from 1-forms to 1-forms. In this case (1) takes the form

$$\begin{cases} \delta(\gamma du) = 0 \text{ in } M \\ u|_{\partial M} = f \end{cases} \tag{15}$$

where $d$ denotes differentiation and $\delta$ codifferentiation with respect to the metric $g$.

The DN map is given by the 1-form

$$\Lambda_{g,\gamma} f = \gamma du|_{\partial M}. \tag{16}$$

An argument similar to the one outlined above shows that

$$\Lambda_{g,\psi_*\gamma} = \Lambda_\gamma \tag{17}$$

for every diffeomorphism $\psi : \overline{M} \to \overline{M}$ which is the identity at the boundary. Here $\psi_*\gamma$ denotes the push-forward by the diffeomorphism $\psi$ of the one form $\gamma$. We remark that Riemannian metrics pull-back naturally under smooth maps and conductivities push-forward naturally under smooth maps.

Now we are in position to state the main conjectures.

**Conjecture A** ($n \geq 3$).

Let $(M, g)$ be a compact Riemannian manifold with boundary. The pair $(\partial M, \Lambda_g)$ determines $(M, g)$ uniquely. Of course uniquely means up to an isometric copy.

**Conjecture B** ($n = 2$).

Let $(M, g)$ be a compact Riemannian surface. Then the pair $(\partial M, \Lambda_g)$ determines uniquely the conformal class of $(M, g)$. Uniquely means again up to an isometric copy.

**Conjecture C** ($n = 2$)

Let $(M, g)$ be a compact Riemannian surface with boundary and $\gamma$ a positive definite symmetric map from 1-forms to 1-forms on $M$. Suppose we know $(M, g, \partial M, \Lambda_{g,\gamma})$ with $\Lambda_{g,\gamma}$ defined as in (16), then we can recover uniquely $\gamma$. Uniquely means here up to an isometry which is the identity on the boundary as in (6)

A basic result which is used in all the anisotropic results stated below is the following Lemma proved in [12]:

**Lemma 1.1.** *(a) $n \geq 3$. Let $(M, g)$ be a compact Riemannian manifold with boundary. Then $\Lambda_g$ determines the $C^\infty$-jet of the metric at the boundary in the following sense. If $g'$ is another Riemannian metric on $M$ such that $\Lambda_g = \Lambda_{g'}$, then there exists a diffeomorphism $\varphi : M \to M$, $\varphi|_{\partial M} =$* Identity *such that $g' = \varphi^* g$ to infinite order at $\partial M$.*

*(b) $n = 2$. Let $(M, g)$ be a compact Riemannian manifold with boundary, then $\Lambda_g$ determines the conformal class of the $C^\infty$-jet of the metric at the boundary.*

*(c) $n = 2$. Let $(M, g)$ be a compact Riemannian surface with boundary. Let $\gamma$ be a positive definite symmetric map from 1-forms to 1-forms. Then the mapping $\Lambda_{g,\gamma}$, as defined in (6), determines the $C^\infty$-jet of the map $\gamma$ at the boundary in the following sense: If $\gamma'$ is another such positive definite symmetric map such that $\Lambda_{g,\gamma} = \Lambda_{g,\gamma'}$. Then there exists a diffeomorphism $\varphi : M \to M$, $\varphi|_{\partial M} =$* Identity *such that $\gamma' = \varphi_* \gamma$ to infinite order at $\partial M$.*

In other words Lemma 1.1 shows that Conjectures A, B, C above are valid at the boundary. The proof of this result is done in case a) by showing that $\Lambda_g$ is a pseudodifferential operator of order 1. Its full symbol, calculated in appropriate coordinates, determines the $C^\infty$-jet of the metric $g$ at the boundary. The proofs of b) and c) are similar.

Lassas and the author proved Conjecture A in the real-analytic case and Conjecture B in general [12]. Moreover these results assume that $\Lambda_g$ is measured only on an open subset of the boundary.

Let $\Gamma$ be an open subset of $\partial M$. We define for $f$, $\operatorname{supp} f \subseteq \Gamma$

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)\Big|_{\Gamma}.$$

The first result of [12] is:

**Theorem 1.2** ($n \geq 3$). *Let $(M, g)$ be a real-analytic compact, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$ be real-analytic and assume that $g$ is real-analytic up to $\Gamma$. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely $(M, g)$.*

Notice that Theorem 1.2 doesn't assume any condition on the topology of the manifold except for connectedness. An earlier result of [14] assumed that $(M, g)$ was strongly convex and simply connected and $\Gamma = \partial M$.

The second result of [12] is the proof of Conjecture B assuming we only measure the DN map on an open subset of the boundary.

**Theorem 1.3** ($n = 2$). *Let $(M, g)$ be a compact Riemannian surface with boundary. Let $\Gamma \subseteq \partial M$ be an open subset. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely the conformal class of $(M, g)$.*

**Sketch of proof of Theorem 1.2.** M. Taylor simplified the arguments in the proof of this result used in [12]. His proof follows the same basic idea of [12] but avoids the use of sheafs in the details of the endgame of the proof. Using this method we were able to extend Theorem 1.2 to the case of complete Riemannian manifolds with boundary [13]. We include Taylor's proof of Theorem 1.2 below [20].

Let us assume that we have two Riemannian manifolds $(M_j, g_j), j = 1, 2$ satisfying the conditions of Theorem 1.2 with $\Gamma \subset \partial M_j$ open and real-analytic and $\Lambda_{g_1}|_\Gamma = \Lambda_{g_2}|_\Gamma$.

Using Lemma 1.1 we can find extensions $\widetilde{M}_j$ of $M_j$ across $\Gamma$, with real-analytic metrics, and there are neighborhoods $\mathcal{O}_j$ of $\Gamma$ in $\widetilde{M}_j$ that are isometric; write $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$. We write $\widetilde{M}_j = M_j \cup \mathcal{O}_j$, with closure $\widehat{M}_j = \overline{M}_j \cup \overline{\mathcal{O}}_j$. Let us set

$$U_j = \widetilde{M}_j \setminus \overline{M}_j, \quad U_1 = U_2 = U. \tag{18}$$

The key point is to use the Dirichlet Green's kernel of $\widetilde{M}_j$ with poles at points in $U$ to construct the desired isometry between $M_1$ and $M_2$.

For $y \in U$, solve

$$\Delta_{g_j} G_{g_j}(\cdot, y) = -\delta_y, \quad G_j(x,y) = 0 \;\text{ for }\; x \in \widetilde{M}_1 = \widetilde{M}_2. \tag{19}$$

We obtain maps

$$\mathcal{G}_j : \widetilde{M}_j \longrightarrow H^s(U), \quad \text{for any } s < 2 - n/2, \tag{20}$$

defined by

$$\mathcal{G}_j(x)(y) = G_j(x,y), \quad x \in \widetilde{M}_j,\ y \in U. \tag{21}$$

If $s < 1 - n/2$ the maps $\mathcal{G}_j$ are of class $C^1$. Furthermore, these maps are real-analytic on the subsets $M_j$. Note that

$$D\mathcal{G}_j(x) : T_x\widetilde{M}_j \longrightarrow H^s(U) \tag{22}$$

is given by

$$D\mathcal{G}_j(x)v = v \cdot \nabla_x G_j(x, \cdot). \tag{23}$$

**Lemma 1.4.** *The map $D\mathcal{G}_j(x)$ is injective for each $x \in \widetilde{M}_j$.*

*Proof.* In fact, if this map annihilates a nonzero $v \in T_x\widetilde{M}_j$, then $v \cdot \nabla_x G_j(x,y) = 0$ for all $y \in U$. This implies by real-analicity $v \cdot \nabla_x G_j(x,y) = 0$ for all $y \in \widetilde{M}_j \setminus \{x\}$. Taking $y = x + \varepsilon v$ (using some local coordinate system) and noting the asymptotic behavior of $G_j(x, x + \varepsilon v)$, we obtain a contradiction. □

From this we obtain:

**Proposition 1.5.** *The map $\mathcal{G}_j : \widetilde{M}_j \to H^s(U)$ is an embedding.*

*Proof.* It remains to show that $x_1 \neq x_2$ in $\widetilde{M}_j \Rightarrow \mathcal{G}_j(x_1) \neq \mathcal{G}_j(x_2)$. If not, then

$$G_j(x_1, y) = G_j(x_2, y) \tag{24}$$

for all $y \in U$, hence, by analyticity, for all $y \in \widetilde{M}_j$. But $G_j(x_1, \cdot)$ is singular only at $y = x_1$, so this gives $x_1 = x_2$. □

The crucial result is the following.

**Proposition 1.6.** *With the identifications $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$ and $U_1 = U_2 = U$, we have*

$$\mathcal{G}_1 = \mathcal{G}_2 \quad on \;\; \mathcal{O}. \tag{25}$$

*Proof.* What we are claiming is that

$$G_1(x,y) = G_2(x,y), \quad \forall\, x \in \mathcal{O},\ y \in U. \tag{26}$$

To see this, fix $y \in U$ and solve

$$\Delta_{g_2} V = 0 \;\text{ on }\; M_2, \quad V(x) = G_1(x,y) \;\text{ for }\; x \in X = M_2. \tag{27}$$

The Dirichlet-to-Neumann hypothesis implies

$$\nabla_x V(x) = \nabla_x G_1(x, y), \quad \text{for} \;\; x \in \Gamma, \tag{28}$$

which in turn, by the unique continuation principle, gives

$$V = G_1(\cdot, y) \;\; \text{on} \;\; \mathcal{O} \cap M_2 = \mathcal{O} \cap M_1, \tag{29}$$

This implies $V$ extends to $\widetilde{M}_2 \setminus \{y\}$, and we have $V(x) = G_2(x, y)$, giving (26). □

We aim to establish the following, which will imply Theorem 1.2.

**Proposition 1.7.** *The sets $\mathcal{G}_1(\widetilde{M}_1)$ and $\mathcal{G}_2(\widetilde{M}_2)$ are identical subsets of $H^s(U)$.*

To show that $\mathcal{G}_1(\widetilde{M}_1) \subset \mathcal{G}_2(\widetilde{M}_2)$, let $B_1$ be the set of points $x \in \widetilde{M}_1$ such that $\mathcal{G}_1(x) \in \mathcal{G}_2(\widetilde{M}_2)$, let $C_1$ be the interior of $B_1$ in $\widetilde{M}_1$, and let $D_1$ be the closure of $C_1$ in $\widetilde{M}_1$. By Proposition 1.6, $D_1$ is not empty. It suffices to show that $D_1$ is open in $\widetilde{M}_1$. As a first step, we have:

**Lemma 1.8.** *Given $x_1 \in D_1$, there exists $x_2 \in \widetilde{M}_2$ such that $\mathcal{G}_2(x_2) = \mathcal{G}_1(x_1)$.*

*Proof.* We know there exist $p_j \in \widetilde{M}_1$, $q_j \in \widetilde{M}_2$ such that $p_j \to x_1$ and $\mathcal{G}_2(q_j) = \mathcal{G}_1(p_j)$. If $\{q_j\}$ has a limit point in $\widetilde{M}_2$, we can denote it $x_2$ and we are done. The only alternative is that $q_j \longrightarrow \widetilde{M}_2$. Then $\mathcal{G}_2(q_j) \to 0$ in $H^s(U)$. This would give $\mathcal{G}_1(p_j) \to 0$ in $H^s(U)$, and hence

$$\mathcal{G}_1(x_1) = 0.$$

But in fact the strong maximum principle gives

$$\mathcal{G}_1(x_1)(y) < 0, \quad \forall\, y \in U,$$

so this shows the alternative is impossible. □

To proceed, we can use the following simple extension of Proposition 1.5.

**Proposition 1.9.** *For each nonempty open set $\Omega \subset U$, the maps*

$$\mathcal{G}_j^{\Omega} : \widetilde{M}_j \longrightarrow H^s(\Omega) \quad (s < 1 - n/2), \tag{30}$$

*given by composing (21) with the operation of restriction to $\Omega$, are embeddings. These maps are real-analytic on $\widetilde{M}_j \setminus \overline{\Omega}$.*

Note that, given $x_j \in \widetilde{M}_j$, we have

$$\mathcal{G}_1(x_1) = \mathcal{G}_2(x_2) \Longleftrightarrow \mathcal{G}_1^{\Omega}(x_1) = \mathcal{G}_2^{\Omega}(x_2) \tag{31}$$

where $T_u$ denotes the tangent space at $u$.

Now let us get back to $x_1 \in D_1$. We have $x_2 \in \widetilde{M}_2$ with $\mathcal{G}_2(x_2) = \mathcal{G}_1(x_1)$, hence

$$\mathcal{G}_2^{\Omega}(x_2) = \mathcal{G}_1^{\Omega}(x_1) = u. \tag{32}$$

Pick $\Omega$ disjoint from $x_1$ (in $\widetilde{M}_1$) and from $x_2$ (in $\widetilde{M}_2$), so $\mathcal{G}_j^{\Omega}$ is an analytic embedding in a neighborhood of $x_j$. Note that

$$T_u \mathcal{G}_1^{\Omega}(x_1) = T_u \mathcal{G}_2^{\Omega}(x_2) = \mathcal{V} \tag{33}$$

is a finite-dimensional subspace of the Hilbert space $H^s(\Omega)$. Let $\mathcal{L}$ denote a linear subspace complementary to $\mathcal{V}$ (e.g., the orthogonal complement of $\mathcal{V}$ with respect to a convenient inner product on $H^s(\Omega)$). By the Implicit Function Theorem, $\mathcal{G}_1^{\Omega}(\widetilde{M}_1)$ and $\mathcal{G}_2^{\Omega}(\widetilde{M}_2)$ are, near $u$, locally graphs of real-analytic functions

$$\Phi_j : \mathfrak{A} \longrightarrow \mathcal{L}, \tag{34}$$

where $\mathfrak{A}$ is an open set in $\mathcal{V}$. Say $u = (u_0, u_1) \in \mathcal{V} \oplus \mathcal{L}$. Then, by the definition of $D_1$, $u_0$ is on the boundary of an open set on which $\Phi_1 = \Phi_2$, so it follows that $\Phi_1 = \Phi_2$ on $\mathfrak{A}$.

Consequently any $x_1 \in D_1$ has a neighborhood whose image under $\mathcal{G}_1^{\Omega}$ is contained in the image of $\mathcal{G}_2^{\Omega}$. In view of (31) this implies $D_1$ is open in $\widetilde{M}_1$. This proves Proposition 1.7 and hence Theorem 1.2. As for Conjecture C the only known result is the case when $M = \Omega$ is an open subset of $\mathbb{R}^n$ with smooth boundary and $g = (\delta_{ij}) =: e$ is the Euclidean metric. More precisely we have

**Theorem 1.10** **($n = 2$).** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $\gamma_1, \gamma_2$ be two anisotropic conductivities so that*

$$\Lambda_{e,\gamma_1} = \Lambda_{e,\gamma_2}.$$

*Then there exists $\psi : \overline{\Omega} \to \overline{\Omega}$ diffeomorphism with $\psi|_{\partial\Omega} =$ Identity so that*

$$\psi_* \gamma_1 = \gamma_2.$$

The proof of Theorem 1.10 follows from a combination of the results of [15] and [17]. In [15] it was proven Theorem 1.10 for isotropic conductivities. One then uses the results of [17] to reduce the anisotropic case to the isotropic one by using the analog of isothermal coordinates in this case. The result is that given an anisotropic conductivity, we can find a diffeomorphism $\phi$ so that $\phi_* \gamma$ is isotropic. We end by mentioning that the result of [15] uses complex geometrical solutions, which will be discussed in the next section.

for all complex frequencies $\rho \in \mathbb{C}^n - 0$, $\rho \cdot \rho = 0$, not just large frequencies. For another construction of these solutions which allow Lipschitz conductivities see [3] (the result of [15] works for $C^2$ conductivities). Theorem 1.10 has been extended to anisotropic non-linear conductivities in [16]. Alessandrini and Barrugo have studied a special case of anisotropy with piecewise analytic conductivities [1].

## 2 The Cauchy Data for the Schrödinger Equation

The only case of Conjecture A that has been settled in general is the isotropic case in Euclidean space. Namely we have in the case that $M = \Omega$ an open, bounded subset of $\mathbb{R}^n$ with a smooth boundary and the metric $g$ is given by

$$g_{ij} = \alpha(x)\delta_{ij}, \qquad \alpha > 0 \tag{35}$$

where $\delta_{ij}$ is the Kronecker delta. The following result was proven in [18].

**Theorem 2.1.** *Let $\Omega \subseteq \mathbb{R}^n$ $n \geq 3$ be a bounded domain with smooth boundary. Let $g^{(i)}$, $i = 1, 2$ be two isotropic Riemannian metrics satisfying (35). Then $\Lambda_{g_1} = \Lambda_{g_2}$ implies $g_1 = g_2$.*

The proof of this result proceeds by proving a more general result by reducing the problem to consider the set of Cauchy data for solutions of the Schrödinger equation (see [21] for more details).

Let $n \geq 3$. Let $q \in L^\infty(\Omega)$. We define the set of **Cauchy data** for the associated Schrödinger equation by

$$\mathcal{C}_q = \left\{ \left( u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \mid (\Delta - q)u = 0 \text{ on } \Omega,\ u \in H^1(\Omega) \right\}. \tag{36}$$

**Theorem 2.2.** *Let $q_i \in L^\infty(\Omega), i = 1, 2$. Assume*

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2}.$$

*Then $q_1 = q_2$.*

The proof of this result uses complex geometrical optics solutions of the Schrödinger equation. Let $q \in L^\infty(\mathbb{R}^n), n \geq 2$ have compact support. Then for $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$, $|\rho|$ sufficiently large, one can construct solutions to

$$(\Delta - q)u_\rho = 0$$

of the form

$$u_\rho = e^{\langle x, \rho \rangle}(1 + \psi_q(x, \rho)) \tag{37}$$

with

$$\|\psi_q(\,\cdot\,, \rho)\|_{H^s(\Omega)} \leq \frac{C}{|\rho|^{1-s}}, \quad 0 \leq s \leq 1, \tag{38}$$

for some $C > 0$ independent of $\rho$.

The function $\psi_q(x, \rho)$ solves

$$\Delta_\rho \psi_q = q(1 + \psi_q), \tag{39}$$

where

$$\Delta_\rho(u) = e^{-\langle x,\rho\rangle} \Delta(e^{\langle x,\rho\rangle} u).$$

The Schwartz kernel $G_\rho$ of $\Delta_\rho^{-1}$ is the so-called Faddeev Green's kernel [7]. The following estimate was proved in [19] ($n = 2$), [18] ($n \geq 3$) for $-1 < \delta < 0$ and $\rho \in \mathbb{C}^n - 0, \rho \cdot \rho = 0$:

$$\|G_\rho f\|_{H^s_\delta} \leq C \frac{\|f\|_{L^2_{\delta+1}}}{|\rho|^{1-s}}. \tag{40}$$

Here $H^s_\alpha$ denotes the Sobolev space associated to the weighted $L^2$ space with norm given by

$$\|f\|^2_{L^2_\alpha} = \int |f(x)|^2 (1 + |x|^2)^\alpha dx.$$

A natural question is whether one can determine the potential by measuring the Cauchy data on strict subsets of the boundary. The only result known beyond the case of real-analytic potentials was proven in [4]. We describe the result below.

We first modify the set of Cauchy data to allow for more singular distributions on the boundary. We define the function space

$$H_\Delta(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid u \in L^2(\Omega),\ \Delta u \in L^2(\Omega)\};$$

$H_\Delta(\Omega)$ is a Hilbert space with the norm

$$\|u\|^2_{H_\Delta(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}.$$

For $u \in H_\Delta(\Omega)$, we have $u|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} \in H^{-\frac{3}{2}}(\partial\Omega)$. We define the set of modified Cauchy data for $q \in L^\infty(\Omega)$ by

$$C_q = \left\{ \left( u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \in H^{-\frac{1}{2}}(\partial\Omega) \times H^{-\frac{3}{2}}(\partial\Omega) \,\Big|\, (\Delta - q)u = 0 \text{ in } \Omega,\ u \in H_\Delta(\Omega) \right\}.$$

If 0 is not a Dirichlet eigenvalue of $\Delta - q$ in $\Omega$ then $C_q$ contains the graph of the Dirichlet-to-Neumann map $\Lambda_q$ conventionally defined on $H^{1/2}(\partial\Omega)$ by the relation $\Lambda_q(f) = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$, where $u \in H^1(\Omega)$ is a solution to the problem

$$(\Delta - q)u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = f;$$

i.e., $\{(f, \Lambda_q(f)) \mid f \in H^{1/2}(\partial\Omega)\} \subset C_q$.

Fix $\xi \in S^{n-1} = \{\xi \in \mathbb{R}^n, |\xi| = 1\}$. We define

$$\partial\Omega_+(\xi) = \{x \in \partial\Omega \mid \langle \nu, \xi\rangle > 0\}, \quad \partial\Omega_-(\xi) = \{x \in \partial\Omega \mid \langle \nu, \xi\rangle < 0\} \tag{41}$$

and for $\varepsilon > 0$

$$\partial\Omega_{+,\varepsilon}(\xi) = \{x \in \partial\Omega \mid \langle \nu, \xi\rangle > \varepsilon\}, \quad \partial\Omega_{-,\varepsilon}(\xi) = \{x \in \partial\Omega \mid \langle \nu, \xi\rangle < \varepsilon\}. \tag{42}$$

We also define the set of restricted Cauchy data

$$C_{q,\varepsilon} = \left\{ \left( u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega_{-,\varepsilon}(\xi)} \right) \Big| \ (\Delta - q)u = 0 \text{ in } \Omega, \ u \in H_\Delta(\Omega) \right\}.$$

The main result of [4] is

**Theorem 2.3.** *Let $n \geq 3$ and $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Given $\xi \in S^{n-1}$ and $\varepsilon > 0$, assume that $C_{q_1,\varepsilon} = C_{q_2,\varepsilon}$. Then $q_1 = q_2$.*

Theorem 2.3 has an immediate consequence in Electrical Impedance Tomography. We assume here now $\gamma$ is an isotropic conductivity, i.e, $\gamma_{ij} = \gamma(x)\delta_{ij}$ with $\gamma \in C^2(\overline{\Omega})$ is a strictly positive function on $\overline{\Omega}$. The Dirichlet-to-Neumann map is defined in this case as follows:

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega}$$

where $u$ solves

$$\operatorname{div} \gamma \nabla u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f.$$

It is easy to see that $\Lambda_\gamma$ extends to a bounded map

$$\Lambda_\gamma : H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{3}{2}}(\partial\Omega).$$

As a direct consequence of Theorem 2.3 we prove

**Corollary 2.4.** *Let $\gamma_i \in C^2(\overline{\Omega})$, $i = 1, 2$, be strictly positive. Given $\xi \in S^{n-1}$ and $\varepsilon > 0$, assume that*

$$\Lambda_{\gamma_1}(f)|_{\partial\Omega_{-,\varepsilon}(\xi)} = \Lambda_{\gamma_2}(f)|_{\partial\Omega_{-,\varepsilon}(\xi)} \quad \forall f \in H^{-\frac{1}{2}}(\partial\Omega).$$

*Then $\gamma_1 = \gamma_2$.*

As far as we know, Theorem 2.3 (Corollary 2.4) is the first global uniqueness result for the Schrödinger equation (conductivity equation) in which the Cauchy data are given only on part of the boundary, beyond the case of a real-analytic potential.

A natural way to attack the problem of finding a potential from partial information of the Cauchy data is to construct solutions of the form (37) with $\psi_q = 0$ on part of the boundary. As it is shown in [9] it is impossible in general to solve the Dirichlet problem for (39) with $\psi_\rho$ decaying (or even polynomially bounded in $\rho$.) In [4] it is shown that we can prescribe Dirichlet conditions for $\psi_\rho$ on particular subsets of the boundary. More precisely we have

**Lemma 2.5.** *Let $n \geq 2$. Let $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$ and $\rho = \tau(\xi + i\eta)$ with $\xi, \eta \in S^{n-1}$. Suppose that $f(\cdot, \rho/|\rho|) \in W^{2,\infty}(\Omega)$ satisfies $\partial_\xi f = \partial_\eta f = 0$, where $\partial_\xi$ denotes the directional derivative in the direction $\xi$. Then we can find solutions to $(\Delta - q)u = 0$ in $\Omega$ of the form*

$$u(x, \rho) = e^{\langle x, \rho \rangle}\Big(f\Big(x, \frac{\rho}{|\rho|}\Big) + \psi(x, \rho)\Big), \quad \psi|_{\partial\Omega_-(\xi)} = 0,$$

*with*

$$\|\psi(\,\cdot\,, \rho)\|_{L^2(\Omega)} \leq \frac{C}{\tau}, \quad \tau \geq \tau_0,$$

*for some $C > 0$ and $\tau_0 > 0$.*

The proof of Theorem 2.3 and Lemma 2.5 uses Carleman estimates for the operator $\Delta_\rho$, which is not an elliptic operator if we consider the dependence on the parameter $\rho$, to construct the solutions and prove the main result. The use of a linear phase function in these Carleman estimates gives rise to the restriction on measuring the Cauchy data on particular subsets of the boundary.

**Theorem 2.6.** *For $q \in L^\infty(\Omega)$ there exist $\tau_0 > 0$ and $C > 0$ such that for all $u \in C^2(\bar{\Omega})$, $u|_{\partial\Omega} = 0$, and $\tau \geq \tau_0$ we have the estimate*

$$\begin{aligned}\tau^2 \int_\Omega |e^{-\tau\langle x,\xi\rangle} u|^2\, dx + \tau \int_{\partial\Omega_+} \langle \xi, \nu\rangle |e^{-\tau\langle x,\xi\rangle} \partial_\nu u|^2\, dS \\ \leq C\left(\int_\Omega |e^{-\tau\langle x,\xi\rangle}(\Delta - q)u|^2\, dx - \tau \int_{\partial\Omega_-} \langle \xi, \nu\rangle |e^{-\tau\langle x,\xi\rangle} \partial_\nu u|^2\, dS\right).\end{aligned}$$

**Sketch of the Proof of Theorem 2.3**

As before we let $\xi \in S^{n-1}$. Fix $k \in \mathbb{R}^n$ such that $\langle \xi, k\rangle = 0$. Using Lemma 2.5, we choose a solution $u_2 \in H_\Delta(\Omega)$ to $(\Delta - q_2)u_2 = 0$ in $\Omega$ of the form

$$u_2 = e^{\langle x, \rho_2\rangle}(1 + \psi_{q_2}(x, \rho_2))$$

with

$$\rho_2 = \tau\xi - i\frac{k + l}{2},$$

where $\langle l, k\rangle = \langle l, \xi\rangle = 0$ and $|k + l|^2 = 4\tau^2$ (with these conditions $\rho_2 \cdot \rho_2 = 0$). In dimension $n \geq 3$ we can always choose such a vector $l$. Since $C_{q_1,\varepsilon} = C_{q_2,\varepsilon}$, there is a solution $u_1 \in H_\Delta(\Omega)$ to $(\Delta - q_1)u_1 = 0$ in $\Omega$ such that

$$u_1|_{\partial\Omega} = u_2|_{\partial\Omega}, \quad \frac{\partial u_1}{\partial\nu}\Big|_{\partial\Omega_{-,\varepsilon}(\xi)} = \frac{\partial u_2}{\partial\nu}\Big|_{\partial\Omega_{-,\varepsilon}(\xi)}.$$

Let us denote $u := u_1 - u_2$ and $q := q_1 - q_2$. We have

$$(\Delta - q_1)u = qu_2 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

It is easy to see that $u|_{\partial\Omega} = 0$ and $u \in H_\Delta(\Omega)$ implies that $u \in H^2(\Omega)$. Also Green's formula is valid for $v \in H_\Delta(\Omega)$. Thus we obtain

$$\int_\Omega (\Delta - q_1)u\bar{v}\,dx = \int_\Omega q u_2 \bar{v}\,dx = \int_\Omega u(\Delta - q_1)\bar{v}\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial\nu}\bar{v}\,dS; \tag{43}$$

Now, we choose

$$\bar{v} = e^{\langle x,\rho_1\rangle}(1 + \psi_{q_1}(x,\rho_1))$$

as in (36) to be a solution to $(\Delta - q_1)\bar{v} = 0$, where

$$\rho_1 = -\tau\xi - i\frac{k-l}{2}$$

with $\xi$, $k$, and $l$ as before so that $\rho_1 \cdot \rho_1 = 0$. Notice that with this choice of $\rho_j$, $j = 1, 2$, we have

$$\rho_1 + \rho_2 = -ik.$$

With these choices of $u_2$ and $v$, the identity (51) now reads

$$\int_\Omega q u_2 \bar{v} = \int_{\partial\Omega} \frac{\partial u}{\partial\nu}\bar{v}\,dS. \tag{44}$$

The final step in the proof is to show that the right hand side of (44) goes to 0 as $\tau \to \infty$.

By hypothesis,

$$\partial_\nu u|_{\partial\Omega_{-,\varepsilon}(\xi)} = 0.$$

Then we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial\nu}\bar{v}\,dS = \int_{\partial\Omega\setminus\partial\Omega_{-,\varepsilon}} \frac{\partial u}{\partial\nu}\bar{v}\,dS = \int_{\partial\Omega_{+,\varepsilon}} \frac{\partial u}{\partial\nu}\bar{v}\,dS.$$

The Cauchy–Schwarz inequality and the estimate $\|\psi_{q_1}\|_{C(\partial\Omega)} \le C\tau^{1/4}$, which follows from (40) and the Sobolev embedding theorem, yields

$$\begin{aligned}
&\Big|\int_{\partial\Omega} \frac{\partial u}{\partial\nu}\bar{v}\,dS\Big| \\
&\quad = \left|\int_{\partial\Omega_{+,\varepsilon}} \frac{\partial u}{\partial\nu} e^{\langle x,\rho_1\rangle}(1 + \psi_{q_1}(x,\rho_1))\,dS\right| \\
&\quad \le \int_{\langle\xi,\nu\rangle\ge\varepsilon} \left|\frac{\partial u}{\partial\nu} e^{-\tau\langle\xi,x\rangle}(1 + \psi_{q_1}(\,\cdot\,,\rho_1))\right|\,dS \\
&\quad \le C(1+\tau^{1/4})\big(\mathrm{Vol}\;\partial\Omega_{+,\varepsilon}\big)^{1/2}\left(\int_{\langle\xi,\nu\rangle\ge\varepsilon} |e^{-\tau\langle\xi,x\rangle}\partial_\nu u|^2\,dS\right)^{\frac{1}{2}}
\end{aligned} \tag{45}$$

for some $C > 0$. Now we use the Carleman estimate of Theorem 2.6 to obtain

$$\begin{aligned}\tau\varepsilon\int_{\partial\Omega_{+,\varepsilon}}|e^{-\tau\langle\xi,x\rangle}\partial_\nu u|^2\,dS &\le \tau\int_{\partial\Omega_+}\langle\xi,x\rangle|e^{-\tau\langle\xi,x\rangle}\partial_\nu u|^2\,dS\\ &\le \int_\Omega |e^{-\tau\langle\xi,x\rangle}(\Delta-q_1)u|^2\,dx\\ &= \int_\Omega |e^{-\tau\langle\xi,x\rangle}qu_2|^2\,dx\\ &\le 2\big(\|q_1\|_{L^\infty(\Omega)}+\|q_2\|_{L^\infty(\Omega)}\big)^2\big(1+\|\psi_2\|^2_{L^2(\Omega)}\big).\end{aligned} \tag{46}$$

Hence, we have proved that

$$\left|\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\bar v\,dS\right| \le C\tau^{-1/4}\to 0,\quad \tau\to\infty.$$

Now letting $\tau\to\infty$ gives

$$\int_\Omega e^{-i\langle x,k\rangle}q(x)\,dx = 0$$

for all $k\perp\xi$. Changing $\xi\in S^{n-1}$ in a small conic neighborhood and using the fact that $\widehat{q}(k)$ is analytic we get that $q=0$ finishing the proof of Theorem 2.3.

**Sketch of Proof of Corollary 2.4.**

It is well known that we can reduce the problem to the case of the Schrödinger equation using the transformation $w=\gamma^{\frac12}u$. If $u$ solves (42), then $w$ solves

$$(\Delta-q)w=0 \text{ in } \Omega$$

with $q=\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. It is easy to see that

$$\Lambda_q(f)=\gamma^{-\frac12}|_{\partial\Omega}\Lambda_\gamma\big(\gamma^{-\frac12}|_{\partial\Omega}f\big)+\frac12\big(\gamma^{-1}\frac{\partial\gamma}{\partial\nu}\big)\big|_{\partial\Omega}f.$$

Now Kohn and Vogelius showed in [10] that given any open subset $\Gamma$ of $\partial\Omega$, if we know $\Lambda_\gamma(f)|_\Gamma$ for all $f$ then we can determine $\gamma|_\Gamma$ and $\frac{\partial\gamma}{\partial\nu}|_\Gamma$, reducing therefore the proof of Corollary 2.4 to Theorem 2.3.

**Conjecture D**

It is natural to expect that one needs to only measure the following subset of the Cauchy data to recover the potential. Let $\Gamma$ be an arbitrary open subset of the boundary. We define

$$C_{q,\Gamma}=\left\{\left(u|_{\partial\Omega},\frac{\partial u}{\partial\nu}\Big|_\Gamma\right)\;\Big|\;(\Delta-q)u=0 \text{ in } \Omega,\ u\in H_\Delta(\Omega)\right\}. \tag{47}$$

The conjecture is that if we know $C_{q,\Gamma}$ then we can recover the potential $q$. It would also be interesting to prove stability estimates and give a reconstruction of the potential under the conditions of Theorem 2.3.

## 3 Semiclassical Complex Geometrical Solutions

As it was noted in the previous section we cannot solve the Dirichlet problem for $\psi_\rho$ satisfying (39) with polynomial control on the growth of $\psi$ for $\rho$ large. However, in [6] it is shown that we can construct approximate complex geometrical solutions of the Schrödinger equation concentrated near planes for large complex frequencies. In some sense these are analog of Gaussian beams for the case of standard geometrical optics and they can be considered as "semiclassical solutions" for the complex principal type operator $\Delta_\rho$. It is also shown in [6] that by measuring the Cauchy data of these approximate solutions on a neighborhood of the intersection of a plane and the boundary one can determine the two plane transform of the potential [8]. We define by

$$Z = \{\rho \in \mathbb{C}^n - 0 : \rho \cdot \rho = 0\},$$

the (complex) characteristic variety of $\Delta$. Each $\rho \in Z$ can be written as $\rho = |\rho|\frac{\rho}{|\rho|} = \frac{1}{\sqrt{2}}|\rho|(\omega_R + i\omega_I) \in \mathbb{R} \cdot \left(S^{n-1} + iS^{n-1}\right)$, with $\omega_R \cdot \omega_I = 0$. For $\rho \in Z$, let $\Delta_\rho = \Delta + 2\rho \cdot \nabla$. Then

$$\Delta_\rho - q(x) = e^{-\langle x,\rho\rangle}(\Delta - q(x))e^{\langle x,\rho\rangle}, \tag{48}$$

so that, with $v(x) = e^{\langle x,\rho\rangle}u(x)$,

$$(\Delta_\rho - q(x))u(x) = w(x) \Leftrightarrow (\Delta - q(x))v(x) = e^{\langle x,\rho\rangle}w(x) \tag{49}$$

and, in particular, $(\Delta_\rho - q(x))u(x) = 0 \Leftrightarrow (\Delta - q(x))v(x) = 0$.

Now, given a potential $q(x)$ and a two-plane $\Pi$, we will construct an approximate solution $u_{app}$ to the Schrödinger equation supported near $\Pi$. We denote by $d\lambda_\Pi$ two-dimensional Lebesgue measure on $\Pi$. We also recall the definition of the variant of $d\lambda_\Pi$ relative to $\Omega$ for the case that $\partial\Omega$ is $C^1$ and $\Pi$ intersects $\partial\Omega$ transversally (for the general case see [6]),

$$< d\lambda_\Pi^\Omega, f >=< d\lambda_\Pi, f \cdot \chi_\Omega >$$

where $\chi_\Omega$ denotes the characteristic function of $\Omega$.

**Theorem 3.1.** *Let $\Omega$ be a bounded domain with smooth boundary and $q(x) \in H^s(\Omega)$ for some $s > \frac{n}{2}$. Then, for any $0 < \beta < \frac{1}{4}$ fixed, the following holds: $\exists\ \epsilon > 0$ such that, for any $\rho = \frac{1}{\sqrt{2}}|\rho|(\omega_R + i\omega_I) \in Z$ and any two-plane $\Pi$ parallel to $\Pi_0 = span\{\omega_R, \omega_I\}$, we can find an approximate solution $u_{app} = u_{app}(x, \rho, \Pi)$ to $(\Delta_\rho - q(x))u = 0$ satisfying*

$$\|u_{app}\|_{L^2(\mathbb{R}^n)} \leq C, \quad \|u_{app}\|_{L^2(\Omega)} \simeq [\lambda_\Pi^\Omega(\Pi \cap \Omega)]^{\frac{1}{2}} \ as\ |\rho| \to \infty \tag{50}$$

$$\text{supp}\,(u_{app}) \subset \left\{ x \in \mathbb{R}^n :\ dist\ (x, \Pi) \leq \frac{2}{|\rho|^\beta} \right\} \tag{51}$$

*and*

$$\|(\Delta_\rho - q)u_{app}\|_{L^2(\mathbb{R}^n)} \leq \frac{C_\epsilon}{|\rho|^\epsilon}. \tag{52}$$

In fact, as will be seen below, $u_{app} = u_0 + u_1$ with $u_0$ depending only on $\Pi$ and $|\rho|$ and satisfying (50.)

We then modify $u_0$ suitably to satisfy the other conditions.

**Remark** Let $\epsilon$ be as in Theorem 3.1. Let $-1 < \delta < 0$. By using the estimate (40) for Faddeev's Green's kernel we can construct a true solution $u$ of $(\Delta - q)u = 0$ in $\Omega$ of the form

$$u = u_0 + u_1 + u_2 \text{ with } \|u_2\|_{H^s_\delta} \leq C|\rho|^{-1-s\epsilon}, \quad 0 \leq s \leq 1. \tag{53}$$

where the constant $C > 0$ depends only on $\|q\|_{L^\infty}$ and the diameter of $\Omega$.

**Sketch of proof of Theorem 3.1**

We use the rotation invariance of $\Delta$ and the invariance of $Z$ under $S^1 = \{e^{i\theta}\}$, and note that it suffices to treat the case $\rho = |\rho|(\boldsymbol{e_1} + i\boldsymbol{e_2})$, where $\{\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}\}$ is the standard orthonormal basis for $\mathbb{R}^n$. Write $x \in \mathbb{R}^n$ as $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ and similarly $\xi = (\xi', \xi'')$.

If $\Pi$ is parallel to $\text{span}\{\omega_R, \omega_I\} = \text{span}\{\boldsymbol{e_1}, \boldsymbol{e_2}\} = \mathbb{R}^2 \times \{0\}$, then $\Pi = \text{span}\{\boldsymbol{e_1}, \boldsymbol{e_2}\} + (0, x_0'')$ for some $x_0'' \in \mathbb{R}^{n-2}$. Given $|\rho| > 1$ and $x_0'' \in \mathbb{R}^{n-2}$, we will define an approximate solution $u(x, \rho, \Pi)$ to $(\Delta_\rho - q(x))u = 0$ on $\mathbb{R}^n$, of the form $u(x, \rho, \Pi) = u_0(x, \rho, \Pi) + u_1(x, \rho, \Pi)$.

For notational convenience, we will usually suppress the dependence on $\rho$ and $\Pi$ and simply write $u(x) = u_0(x) + u_1(x)$. We will use various cutoff functions $\chi_j$, for $j$ even or odd, $\chi_j$ will always denote a function of $x'$ or $x''$, respectively. Also, $B^m(a; r)$ and $S^{m-1}(a; r)$ will denote the closed ball and sphere of radius $r$ centered at a point $a \in \mathbb{R}^m$.

**Construction of $u_0$**

To define $u_0$, first fix $\chi_0 \in C_0^\infty(\mathbb{R}^2)$ with $\chi_0 \equiv 1$ on $B^2(0; R)$ for any $R > sup\{|x'| : (x', x'') \in \Omega$ for some $x'' \in \mathbb{R}^{n-2}\}$; let $C_0 = \|\chi_0\|_{L^2(\mathbb{R}^2)}$. Secondly, let $\psi_1 \in C_0^\infty(\mathbb{R}^{n-2})$ be radial, non-negative, supported in the unit ball, and satisfy

$$\int_{\mathbb{R}^{n-2}} (\psi_1(x''))^2 dx'' = 1.$$

Now, for $\beta > 0$ to be fixed later, we let $\delta$ be the small parameter $\delta = |\rho|^{-\beta}$ and define

$$\chi_1(x'') = \delta^{-\frac{n-2}{2}} \psi_1\left(\frac{x' - x_0''}{\delta}\right),$$

so that

$$\|\chi_1\|_{L^2(\mathbb{R}^{n-2})} = \|\psi_1\|_{L^2(\mathbb{R}^{n-2})} = 1, \; \forall\, \delta > 0. \tag{55}$$

Set $u_0(x) = u_0(x', x'') = \chi_0(x')\chi_1(x'')$; then $u_0$ is real, $||u_0||_{L^2(\mathbb{R}^n)} = C_0$ and $||u_0||_{L^2(\Omega)} \to [\lambda_\Pi(\Pi \cap \Omega)]^{\frac{1}{2}}$ as $\delta \to 0^+$, i.e., as $|\rho| \to \infty$. Note also that $||u_0||_{H^1} \leq c\delta^{-1} = c|\rho|^\beta$, so that $||u_0||_{H^s} \leq c|\rho|^{s\beta}$ for $0 \leq s \leq 1$. Since $\Delta_\rho = \Delta + 2\rho \cdot \nabla = \Delta + 2|\rho|(\boldsymbol{e_1} + i\boldsymbol{e_2}) \cdot \nabla = \Delta + 4|\rho|\bar{\partial}_{x'}$ and $\rho \bot \mathbb{R}^{n-2}$,

$$\begin{aligned}(\Delta_\rho - q(x))u_0 &= (\Delta\chi_0) \cdot \chi_1 + 2(\nabla\chi_0) \cdot (\nabla\chi_1) + \chi_0(\Delta\chi_1) \\ &\quad +2(\rho \cdot \nabla)(\chi_0)\chi_1 + 2\chi_0(\rho \cdot \nabla)(\chi_1) - q\chi_0\chi_1 \\ &= \chi_0(x')(\Delta_{x''} - q)(\chi_1)(x'') \text{ on } B^2(0; R) \times \mathbb{R}^{n-2},\end{aligned}$$

the first and fourth terms after the first equality vanishing because $(\rho \cdot \nabla)(\chi_0) = 2\bar{\partial}\chi_0 \equiv 0$ on $B^2(0; R)$, and the second and fifth equaling zero because $\nabla\chi_1 \bot \mathbb{R}^2$.

**Construction of $u_1$.**

To define the second term in the approximate solution, $u_1(x)$, we make use of a truncated form of the Faddeev Green's function, $G_\rho$, and an associated projection operator. The operator $\Delta_\rho$ has, for $\rho \in Z$, (full) symbol

$$\sigma(\xi) = -[(|\xi|^2 - 2|\rho|\omega_I \cdot \xi) + i2|\rho|(\omega_R \cdot \xi)], \tag{56}$$

and so for $\frac{\rho}{|\rho|} = \boldsymbol{e_1} + i\boldsymbol{e_2}$, we have

$$\sigma(\xi) = -[(|\xi - |\rho|\boldsymbol{e_2}|^2 - |\rho|^2) + i(2|\rho|\xi_1)],$$

which has (full) characteristic variety

$$\begin{aligned}\Sigma_\rho &= \{\xi \in \mathbb{R}^n : \xi_1 = 0, |\xi - |\rho|\boldsymbol{e_2}| = |\rho|\} \\ &= \{0\} \times S^{n-2}((|\rho|, 0, \ldots, 0); |\rho|) \subset \mathbb{R}_{\xi_1} \times \mathbb{R}^{n-1}_{\xi_2, \xi''}.\end{aligned} \tag{58}$$

The Faddeev Green's function is then defined by $G_\rho = (-\sigma(\xi)^{-1})^\vee \in \mathcal{S}'(\mathbb{R}^n)$. We now introduce, for an $\epsilon_0 > 0$ to be fixed later, a tubular neighborhood of $\Sigma_\rho$,

$$T_\rho = \{\xi : (\xi, \Sigma_\rho) < |\rho|^{-\frac{1}{2} - \epsilon_0}\}, \tag{59}$$

as well as its complement, $T_\rho^C$, and let $\chi_{T_\rho}, \chi_{T_\rho^C}$ be their characteristic functions. Define a projection operator, $P_\rho$, and a truncated Green's function, $\widetilde{G}_\rho$, by

$$\widehat{P_\rho f}(\xi) = \chi_{T_\rho}(\xi) \cdot \hat{f}(\xi) \quad \text{and} \tag{60}$$

$$(\widetilde{G}_\rho f)^\wedge(\xi) = \chi_{T_\rho^C}(\xi) \cdot [-\sigma(\xi)]^{-1}\hat{f}(\xi) \tag{61}$$

for $f \in (\mathbb{R}^n)$. Note that $\Delta_\rho \widetilde{G}_\rho = I - P_\rho$.

Choose a $\psi_3 \in C_0^\infty(\mathbb{R}^{n-2})$, supported in $B^{n-2}(0; 2)$, radial and with $\psi_3 \equiv 1$ on supp $(\psi_1)$, and set $\chi_3(x'') = \psi_3(\frac{x'' - x_0''}{\delta})$. We now define the second term, $u_1(x, \rho, \Pi)$ in the approximate solution by

$$u_1(x) = -\chi_3(x'')\widetilde{G}_\rho((\Delta_\rho + q(x))u_0(x)) \tag{62}$$

and set $u(x) = u_0(x) + u_1(x)$. Then $u_1$ (as well as $u_0$) is supported in $\{x : \text{dist}\,(x, \Pi) \le 2\delta\}$, yielding (51). We will see below that $\|u_1\|_{L^2(\Omega)} \le C|\rho|^{-\epsilon}$ as $|\rho| \to \infty$, so that the first part of (50) holds as well. To start the proof of (52), note that

$$\begin{aligned}
&(\Delta_\rho - q)(u_0 + u_1)\\
&\qquad = (\Delta_\rho - q)u_0 - (\Delta_\rho - q)\chi_3 \widetilde{G}_\rho((\Delta_\rho - q)u_0)\\
&\qquad = (\Delta_\rho - q)u_0 - \chi_3(\Delta_\rho - q)\widetilde{G}_\rho((\Delta_\rho - q)u_0)\\
&\qquad\quad - [\Delta_\rho - q, \chi_3]\widetilde{G}_\rho((\Delta_\rho - q)u_0)\\
&\qquad = (\Delta_\rho - q)u_0 - \chi_3(I - P_\rho)(\Delta_\rho - q)u_0 - \chi_3 q\widetilde{G}_\rho(\Delta_\rho - q)u_0\\
&\qquad\quad - 2(\nabla\chi_3 \cdot \nabla_{x''})\widetilde{G}_\rho(\Delta_\rho + q)u_0 - (\Delta_{x''}\chi_3)\widetilde{G}_\rho(\Delta_\rho - q)u_0\\
&\qquad = \chi_3 P_\rho(\Delta_\rho - q)u_0\\
&\qquad\quad - [-q\chi_3 + 2(\nabla\chi_3 \cdot \nabla_{x''}) - (\Delta_{\chi''}\chi_3)]\widetilde{G}_\rho(\Delta_\rho - q)u_0
\end{aligned}$$

on $\Omega$, since $\chi_3 \equiv 1$ on $\text{supp}\,(\chi_1)$. Now, since $q_1\chi_3 \in L^\infty$, $|\nabla\chi_3| \le C\delta^{-1} = c|\rho|^\beta$ and $|\Delta_{x''}\chi_3| \le C\delta^{-2} = c|\rho|^{2\beta}$, (52) will follow if we can show that for some $\epsilon > 0$,

$$\|P_\rho(\Delta_\rho - q)u_0\|_{L^2(\Omega)} \le C|\rho|^{-\epsilon}, \tag{63}$$

$$\|\,|D''|\widetilde{G}_\rho(\Delta_\rho - q)u_0\|_{L^2(\Omega)} \le C|\rho|^{-\beta-\epsilon}, \text{ and} \tag{64}$$

$$\|\widetilde{G}_\rho(\Delta_\rho - q)u_0\|_{L^2(\Omega)} \le C|\rho|^{-2\beta-\epsilon}, \tag{65}$$

with $C$ independent of $|\rho| > 1$. Here $D$" denotes differentiation in the $x$" variables. The details of the proof of these estimates can be found in [6] and we omit then.

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